2026 WAIME Answer Key

1	**077**	6	**750**	11	**005**
2	**114**	7	**079**	12	**603**
3	**153**	8	**106**	13	**243**
4	**083**	9	**576**	14	**289**
5	**123**	10	**429**	15	**040**

(Answer key is for reference; full solutions begin on the next page.)

2026 WAIME Solutions

1. How many possible values of $a! + b! + c!$ have less than 5 digits?

   Proposed by Ryan Tang

   Solution. Note that $8! > 10000$, so $a! + b! + c!$ can only have less than 5 digits if $a, b, c < 7$.
   Note that if two or more of a, b, c equal to 7, then $a! + b! + c! > 7! + 7! > 10000$. So at most one of a, b, c can equal to 7.

   Case 1: One of a, b, c is 7
   Note that because $7! + 6! + 6! < 10000$, all possible values work. So the number of ways is $\binom{6}{2} + 6 = 21$, depending on if the other two numbers are the same or not.
   Case 2: None of a, b, c equal to 7
   Note that all solutions work, since $6! + 6! + 6! < 10000$.
   So the number of solutions is $\binom{6}{3} + 2\binom{6}{2} + 6 = \boxed{077}$, depending on whether all 3 numbers are distinct, if 2 are the same, or if they are all the same. Note that if 2 are the same, then there are two cases based on which number appears twice, so it is multiplied by 2.

2. Let $x > 1$ and $y > 1$ satisfy the system

$$\log_{10}(x^2 - 1) = \log_{10}(y^3 - 1) - 1,$$
$$\log_{10}(y^3 - 1) = \log_{10}(x^4 - 1) - 2.$$

   If $x = a\sqrt{b}$, where a and b are positive integers and b is not divisible by the square of any prime, find $a + b$.

   Proposed by Vivaan Mukherjee

   Solution. From the first equation, $\log_{10}(x^2 - 1) = \log_{10}(\frac{y^3-1}{10})$, so $10x^2 - 10 = y^3 - 1$
   From the second equation, $\log_{10}(y^3 - 1) = \log_{10}(\frac{x^4-1}{100})$, so $100y^3 - 100 = x^4 - 1$
   From the first equation, get that $y^3 = 10x^2 - 9$. Substitute this into the second equation to get $100(10x^2 - 9) - 100 = x^4 - 1$
   $1000x^2 - 1000 = x^4 - 1$
   $x^4 - 1000x^2 + 999 = 0$
   $(x^2 - 1)(x^2 - 999) = 0$
   $x^2 = 999$, so $x = \sqrt{999} = 3\sqrt{111}$, and the answer is $3 + 111 = \boxed{114}$

3. Kaiyuan is pouring 100 cubic units of water into a cup with shape $z = |x| + |y|$. After he pours all the water in, the water reaches height N. N can be expressed as $\sqrt[b]{a}$, where the prime factorization of a does not have a perfect power of b or higher. Find the value of $a + b$.

   Proposed by Ryan Tang

   Solution. Let h be the height of the cup. Note that $|x| + |y| = h$, which forms a square with points $(0, h)$, $(h, 0)$, $(0, -h)$, and $(-h, 0)$
   Note that this square has side length $h\sqrt{2}$, so its area is $2h^2$
   Because the cup is a square pyramid, the volume is $\frac{1}{3} \cdot 2h^2 \cdot h = \frac{2h^3}{3} = 100$
   Solving gives $h = \sqrt[3]{150}$, so the answer is $150 + 3 = \boxed{153}$

4. Let a decimal be "amazing" if it is between 0 and 1 and has three nonzero digits after the decimal point. For instance, 0.351 and 0.989 are amazing, but 0.705 and 1.321 are not amazing. When all amazing decimals are multiplied together, the final result has N digits after the decimal point. Find the remainder when N is divided by 1000.

   Proposed by Ryan Tang

   Solution. Note that there are a total of $9^3 = 729$ amazing decimals.
   So the product of all amazing decimals is $\frac{111 \cdot 112 \cdot \cdots \cdot 999}{10^{3 \cdot 729}}$
   Note that the number of digits after the decimal point is equal to the value of the exponent of this fraction after it is simplified.
   This equals to $3 \cdot 729$ minus the number of trailing zeroes of $111 \cdot 112 \cdot \cdots \cdot 999$
   The number of trailing zeroes is the same as the amount of 5s in its prime factorization, since there are more 2s than 5s, and one 2 and one 5 form one trailing zero.
   So analyze the numbers in the product
   Divisible by 5:
   For a number in this product to be divisible by 5, the unit digit has to be 5, since it cannot be 0. So $9 \cdot 9 \cdot 1 = 81$ numbers satisfy this property.
   Divisible by 25:
   For a number in this product to be divisible by 5, the last two digits have to be 25 or 75. So $9 \cdot 2 = 18$ numbers satisfy this property.
   Divisible by 125:
   This includes $125 \cdot 1$, $125 \cdot 3$, $125 \cdot 5$, and $125 \cdot 7$, so there are 4 numbers that satisfy this.
   Divisible by 625:
   Only 625 satisfies this property.

   So the product has $81 + 18 + 4 + 1 = 104$ powers of 5 in its prime factorization, and the answer is $3 \cdot 729 - 104 = 2\boxed{083}$

5. Let a, b, and c be the roots to $x^3 - mx^2 + nx - 128$ satisfying $a < b < c$ and $b^2 - 4ac = 0$, where m and n are positive integers. What is the minimum possible value of $m + n$?

   Proposed by Ryan Tang

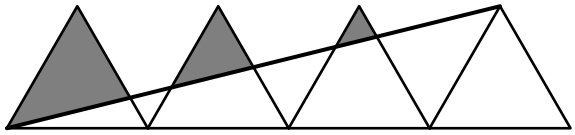
   Solution. By vietas, $a + b + c = m$, $ab + bc + ac = n$, and $abc = 128$
   Note that $b = \sqrt{4ac} = 2\sqrt{ac}$. So $abc = ac \cdot 2\sqrt{ac} = 128$, so $ac\sqrt{ac} = 64$, $(ac)^{\frac{3}{2}} = 64$, and $ac = 64^{\frac{2}{3}} = 16$. This results in $b = 8$
   Note that $n = ab + bc + ac = b(a + c) + ac = 8(a + c) + 16$ and $m = a + b + c = 8 + a + c$
   Note that because m and n are positive integers, $a + c$ is a positive integer
   Because $m + n = 9(a + c) + 24$, the minimum value of $m + n$ happens at the minimum value of $a + c$.
   This happens when a and c are as close as possible, since ac is a fixed value.
   Because $b = 8$, $a < 8 < c$, so the closest a and c can get is when $a < 2$ and $c > 8$, so $a + c > 10$. This results in $a + c = 11$ and $m + n = 9(a + c) + 24 = 9 \cdot 11 + 24 = \boxed{123}$

6. Non-degenerate octagon $ABCDEFGH$ with area 30 satisfies that $ACEG$ is a square and $\angle ABC = \angle CDE = \angle EFG = \angle GHA = 90^\circ$. Square $JKLM$ with side lengths parallel to $ACEG$ is constructed such that points B, D, F, and H lie on the square. The value of $\frac{[JKLM]}{[ACEG]}$ is a positive integer. Find the sum of all possible values of $[ACEG]^2$, given that $ACEG$ is inside $ABCDEFGH$.

Proposed by Ryan Tang

Solution. Let s be the side length of the square and let a, b, c, d be the heights of the 4 triangles outside of the square.
Note that because $BDFH$ lie on a square parallel to $ACEG$, $s + a + c = s + b + d$
Also, note that the area of $ABCDEFG$ is $s^2 + \frac{1}{2}(sa + sb + sc + sd) = s^2 + sa + sc = s(s + a + c) = 30$
Let $p = s + a + c$. Get that $sp = 30$
By Thales's theorem, get that $a, b, c, d \leq \frac{s}{2}$
So $p = s + a + c \leq 2s$ and $s < p \leq 2s$
Also, get that the area of $JKLM$ is $(s + a + c)^2 = p^2$
So $\frac{p^2}{s^2}$ is a positive integer. Because $s < p \leq 2s$, this quotient can equal 2, 3, or 4.
Note that if the quotient equals 4, then the octagon becomes degenerate, since all the triangles have to be isosceles right, which forms a square instead of an octagon.
So $\frac{p^2}{s^2} = 2, 3$
$\frac{(\frac{30}{s})^2}{s^2} = 2, 3$
$\frac{900}{s^4} = 2, 3$
$s^4 = 450, 300$
So the answer is $450 + 300 = \boxed{750}$

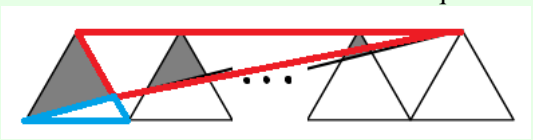
7. 10 congruent equilateral triangles are placed side by side along a straight line, all pointing upward. A line is drawn from the leftmost bottom vertex of the first triangle to the top vertex of the 10th triangle, as shown. In the first 9 triangles, the region above the line and inside the triangle is shaded. If the shaded region has area $\frac{m}{n}$ times the total area of all 10 triangles, where m and n are relatively prime positive integers, find $m + n$.



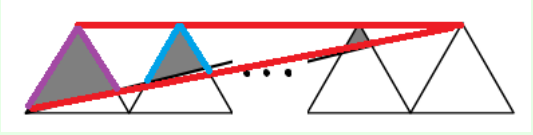
A simplified version of this problem with 4 triangles instead of 10

Proposed by Kaiyuan Mao

Solution. Let A be the area of one equilateral triangle.



Note that the blue and red triangles are similar with ratio $1:9$, so the shaded area in the first triangle is $\frac{9}{10}A$, since the height is the same, and the ratio of the bases are $9:10$



Note that the blue triangle is a $\frac{8}{9}$ dilation of the purple triangle. So the area of the blue triangle is $(\frac{9}{10}A)\cdot(\frac{8}{9})^2$

Use similar logic to get that the total shaded area is $(\frac{9}{10}A)(\frac{9^2}{9^2}+\frac{8^2}{9^2}+\frac{7^2}{9^2}+\cdots+\frac{1^2}{9^2}) = (\frac{9}{10}A)\cdot\frac{\frac{9\cdot10\cdot19}{6}}{9^2} = \frac{19}{6}A$

So the fraction of shaded area is $\frac{\frac{19}{6}A}{10A} = \frac{19}{60}$, and the answer is $19+60=\boxed{079}$

8. Ryan starts with a point at $(2,0)$ in the coordinate plane. Each turn, he rotates the point 180° about one of the points $(0,0)$, $(1,1)$, or $(2,2)$. After exactly 2026 turns, the greatest possible distance between any two attainable final positions can be expressed as $a\sqrt{b}$, where a and b are integers and b is not divisible by the square of any prime, find the remainder when $a+b$ is divided by 1000.

Proposed by Ryan Tang

Solution. Note that if the point is at (a,b), then after one turn, it can become $(-a,-b)$, $(2-a,2-b)$, or $(4-a,4-b)$. After the second turn, it can become $(4+a,4+b),(2+a,2+b),\cdots,(-4+a,-4+b)$. Get that after 2026 turns, it can become $(2\cdot2026+a,2\cdot2026+a),(2\cdot2025+a,2\cdot2025+a),\cdots,(2\cdot(-2026)+a,2\cdot(-2026)+a)$. So the maximum possible distance is $\sqrt{(4\cdot2026)^2+(4\cdot2026)^2} = 8104\sqrt{2}$ and the answer is $8104+2=8\boxed{106}$

9. Let $S=\{1,2,\ldots,14\}$. How many nonempty subsets of S contain no pair of integers who have a sum of 15 or a difference of 1?

Proposed by Unknown

Solution. Consider the pairs $(1,14),(2,13),(3,12),\cdots,(6,9),(7,8)$. Note that for each pair, at most 1 item can be in the subset, or there would be a pair that sums to 15.

Let a_n be the number of ways to choose items from n pairs such that a fixed item is chosen from the last pair. Let b_n be the number of ways to choose items from n pairs such that an item is not chosen from the last pair.

Note that $a_n = a_{n-1}+b_{n-1}$ and $b_n = 2a_{n-1}+b_{n-1}$. Also, note that $a_1=1$ and $b_1=1$

Note that the answer is $2a_7+b_7-1 = 2\cdot169+239-1=\boxed{576}$

10. Let t_n be a sequence defined by $t_0=1$, $t_1=\frac{1}{7}$, $t_2=\frac{1}{4}$, and

$$(4t_{n-3}t_{n-2}-9)(4t_{n-1}t_n-1) = 9-24t_{n-3}t_n.$$

Find the smallest positive integer n such that $t_nt_{n-1} < \frac{1}{3^{434}}$.

Proposed by Wesley Wu

Solution. Let $a_n = t_n t_{n-1}$
$(4a_{n-2} - 9)(4a_n - 1) = 9 - 24\frac{a_n a_{n-2}}{a_{n-1}}$
$16a_n a_{n-2} - 4a_{n-2} - 36a_n + 9 = 9 - 24\frac{a_n a_{n-2}}{a_{n-1}}$
$a_n(16a_{n-2}a_{n-1} - 36a_{n-1}) - 4a_{n-2}a_{n-1} = -24a_n a_{n-2}$
$a_n(4a_{n-2}a_{n-1} + 6a_{n-2} - 9a_{n-1}) = a_{n-2}a_{n-1}$
$a_n = \frac{a_{n-2}a_{n-1}}{4a_{n-2}a_{n-1}+6a_{n-2}-9a_{n-1}}$
Note that $a_1 = \frac{1}{7}$ and $a_2 = \frac{1}{28}$
Get that $a_3 = \frac{1}{109}$
This motivates setting $b_n = \frac{1}{a_n}$. Then, $b_1 = 7$, $b_2 = 28$, $b_3 = 109$. Note that $\frac{1}{b_n} = \frac{\frac{1}{b_{n-2}b_{n-1}}}{\frac{4}{b_{n-2}b_{n-1}}+\frac{6}{b_{n-2}}-\frac{9}{b_{n-1}}}$,
so $b_n = 4 + 6b_{n-1} - 9b_{n-2}$
$b_{n-1} = 4 + 6b_{n-2} - 9b_{n-3}$
Subtraction gives $b_n - b_{n-1} = 6b_{n-1} - 6b_{n-2} - 9b_{n-2} + 9b_{n-3}$, so $b_n - 7b_{n-1} + 15b_{n-2} - 9b_{n-3} = 0$
The characteristic polynomial is $x^3 - 7x^2 + 15x - 9 = 0$
$(x-1)(x-3)^2 = 0$
So $b_n = 1^n \cdot A + 3^n \cdot (Bn + C)$
Plugging in $n = 1, 2, 3$ and solving results in $A = B = C = 1$
So $b_n = 1 + 3^n(n+1)$
This problem asks for the smallest n such that $b_n > 3^{434}$
$1 + 3^n(n+1) > 3^{434}$
Note that the answer should be close to 434 because of the exponent. At $n \approx 434$, $3^5 < n + 1 < 3^6$. So $3^{5+n} < 3^n(n+1) < 3^{6+n}$
So $1 + 3^{5+n} > 3^{434}$ and $n = \boxed{429}$

11. Let the roots of $P(x) = x^5 + 2x^4 - 3x^3 + 2x - 3$ be a, b, c, d, and e. Compute

$$\frac{1}{a^3+1} + \frac{1}{b^3+1} + \frac{1}{c^3+1} + \frac{1}{d^3+1} + \frac{1}{e^3+1}.$$

Proposed by Wesley Wu & Ryan Tang

Solution. Note that $x^5 + 2x^4 - 3x^3 + 2x - 3 = x^3(x^2 + 2x - 3) + 2x - 3 = 0$, so $x^3 = \frac{-2x+3}{x^2+2x-3}$, $x^3 + 1 = \frac{x^2}{x^2+2x-3}$, and $\frac{1}{x^3+1} = \frac{x^2+2x-3}{x^2} = 1 + \frac{2}{x} - \frac{3}{x^2}$
So the answer is $\sum_{cyc}(1 + \frac{2}{a} - \frac{3}{a^2}) = 5 + 2 \cdot \frac{-2}{-3} - 3 \cdot ((\frac{-2}{-3})^2 - 2(\frac{0}{-3})) = 5 + \frac{4}{3} - \frac{4}{3} = \boxed{005}$

Alternate Solution: Let $1, \omega, \omega^2$ be the third roots of unity and then use partial fractions to split the sum.

12. Find the minimum value of k such that

$$\sum_{x=1}^{k}\sum_{m=1}^{3^x} 3^m \prod_{n=1}^{3^x}\left(\left\lfloor\frac{m}{3^n}\right\rfloor - 3\left\lfloor\frac{m}{3^{n+1}}\right\rfloor - 1\right)^2$$

is divisible by 1000.

Proposed by Ryan Tang

Solution. Note that

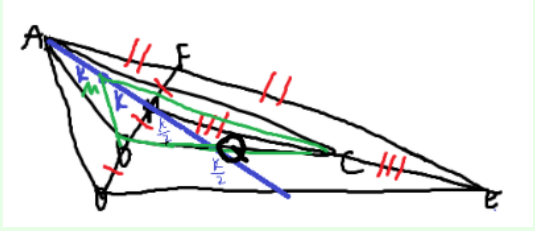
$$\prod_{n=1}^{3^x}\left(\left\lfloor\frac{m}{3^n}\right\rfloor-3\left\lfloor\frac{m}{3^{n+1}}\right\rfloor-1\right)^2=\prod_{n=1}^{3^x}\left(\left\lfloor\frac{m}{3^n}\right\rfloor-3\left\lfloor\frac{\left\lfloor\frac{m}{3^n}\right\rfloor}{3}\right\rfloor-1\right)^2=\prod_{n=1}^{3^x}\left(\left(\left\lfloor\frac{m}{3^n}\right\rfloor \pmod 3\right)-1\right)^2$$

This equals 0 if in the base 3 representation of m, one of the digits that isn't the unit digit equals to 1, since $\lfloor\frac{m}{3^n}\rfloor$ is all the digits up to the nth one, and when taken $\pmod 3$, the bigger digits cancel out, and it just becomes the nth digit.
So this sum becomes $\sum_{x=1}^{k} 3^{1_3}+3^{2_3}+3^{20_3}+3^{21_3}+3^{22_3}+3^{200_3}+\cdots+3^{222\cdots_3}$, where there are x 2s in the last one.
This can be written as $(3^0+3^1+3^2)(3^{0\cdot 3}+3^{2\cdot 3})(3^{0\cdot 9}+3^{2\cdot 9})\cdots(3^{0\cdot 3^{x-1}}+3^{2\cdot 3^{x-1}})-3^0$
Note that $3^{0\cdot 3^{x-1}}+3^{2\cdot 3^{x-1}}=1+9^{3^{x-1}}$
$1+9^{3^{x-1}}\equiv 1-1^{3^{x-1}}\equiv 0 \pmod{10}$. So if $x>3$, the big product would be $\equiv 0 \pmod{1000}$, so the value of the sum would be $\equiv -1 \pmod{1000}$
From the multiplication expression, get that when $x=1,2,3$, the sum equals to $12, 489, 99 \pmod{1000}$.
So when $k=3$ the sum is $600 \pmod{1000}$
After that, increasing k by 1 decreases the sum by 1, since the expression equals $-1 \pmod{1000}$
So the answer is $3+600=\boxed{603}$

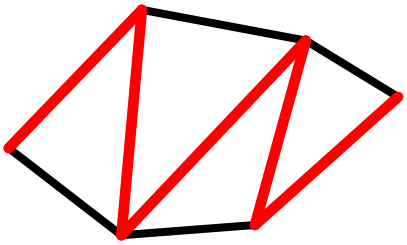
13. Triangle ABC satisfies $AB=3$, $BC=5$, and $AC=7$. A point P lies in the interior of the triangle. Let D be the reflection of P across point B, and let E be the reflection of P across point C. Let F denote the reflection of point B across P. If F is the midpoint of segment AE, determine the square of the area of $\triangle ADE$.

Proposed by Ryan Tang

Solution. Note that $DB=BP=PF$, so $\frac{DP}{PF}=2$. Also, note that F is the midpoint of AE. Because of the $2:1$ ratio, get that P is the centroid of $\triangle ADE$.
Let M be the midpoint of AP
Get that $[MBC]=\frac{[ADE]}{4}$
Let AP intersect BC at Q
Let $AM=MP=k$. Also, note that P is the centroid of $\triangle MBC$, since it is just a dilated version of $\triangle ADE$. So $MP=2PQ$ and $PQ=\frac{k}{2}$.
Also, get that $[ABC]=\frac{15\sqrt{3}}{4}$ through heron's formula.
So $[ABC]=\frac{\frac{5}{2}k}{\frac{3}{2}k}[MBC]=\frac{5}{3}[MBC]=\frac{5}{12}[ADE]=\frac{15\sqrt{3}}{4}$ and $[ADE]=9\sqrt{3}$, so the answer is $(9\sqrt{3})^2=\boxed{243}$



14. A non-degenerate convex hexagon with area 72 has two opposite sides that are equal. Then, 3 diagonals are drawn inside the hexagon such that when combined, they connect one point of one side with the opposite point of the other side. The sum of the lengths of these diagonals and one of the equal sides equals to 24. What is the minimum integer value of the sum of squares of the other 4 sides of the hexagon?



Proposed by Kaiyuan Mao & Ryan Tang

Solution. Let the common side length be a and the lengths of the diagonals be b, c, and d.
Note that $a+b+c+d=24$
Note that the hexagon area is $72=\frac{1}{2}(ab\sin\theta_1+bc\sin\theta_2+cd\sin\theta_3+ad\sin\theta_4)\leq\frac{1}{2}(ab+bc+cd+ad)=\frac{1}{2}(a+c)(b+d)\leq\frac{1}{2}\cdot 12\cdot 12=72$, where $\theta_1,\theta_2,\theta_3$, and θ_4 are the angles formed by the lines
So for this to happen, all the equality cases have to be true, and all the angles are 90°
Then, the sum of the squares of the other side lengths equals $(a^2+b^2)+(b^2+c^2)+(c^2+d^2)+(d^2+a^2)=2(a^2+b^2+c^2+d^2)\leq 2(6^2+6^2+6^2+6^2)=288$. However, the 288 case does not work, since it results in the 6 points forming a quadrilateral instead of a hexagon, which is degenerate. As a result, the answer is $\boxed{289}$.

15. A rectangle of dimensions $4\sqrt{3}$ by 4 has a point P chosen uniformly at random inside it. James folds each vertex onto P, forming four crease lines that partition the rectangle. The expected number of regions formed by those lines is $a-\frac{b\pi}{c\sqrt{d}}$ where a,b,c,d are positive integers, $\gcd(b,c)=1$, and d is squarefree. Find $a+b+c+d$.

Proposed by Wesley Wu

Solution. Claim: Let a line segments be drawn inside a rectangle. Assume no three lines intersect at one point, and let b be the number of intersection points. Then, the rectangle is divided into $a+b+1$ regions. We begin with one total region. Each segment added adds one extra region, and each extra intersection adds one more region. Thus, the claim is proven.

There are four lines inside our rectangle, so we want to calculate the expected value of intersections $+ 5$. Each vertex results in one distinct line.

By linearity of expectation, we can split this into two independent cases–one for adjacent vertices and one for opposite vertices. The total area of the rectangle is $16\sqrt{3}$.

Case 1: Adjacent vertex creases intersect. WLOG consider vertices A and B. Note that since both creases are perpendicular bisectors, their intersection is the circumcenter. Furthermore, the circumcenter is only within the rectangle when the triangle PAB is acute; otherwise, it would be on the opposite side of AB as P (excluding right angle edge cases). This means that the region of success is simply the area outside the semicircle with diameter AB. Generalizing (after verifying that $2\sqrt{3} < 4$), this case gives an expectation of

$$\frac{2((16\sqrt{3}-6\pi)+(16\sqrt{3}-2\pi))}{16\sqrt{3}} = \frac{64\sqrt{3}-16\pi}{16\sqrt{3}} = 4 - \frac{\pi}{\sqrt{3}}.$$

Case 2: Opposite vertex creases intersect. Here, choosing vertices A and C is symmetric to choosing B and D, so we work with PAC.

Since the intersection of the creases is the circumcenter of PAC, it must be equidistant from A and C. The edge case is when the circumcenter actually lies on one of the sides. Thus, there are two successful regions for vertices A and C, both of which are a trapezoid minus a sector. The area of the trapezoid is equal to half that of the rectangle by symmetry, and the sector has angle 60° and radius $\frac{8}{\sqrt{3}}$, for an area of $\frac{32\pi}{9}$. Therefore, each small region has probability $\frac{8}{\sqrt{3}} - \frac{32\pi}{9}$ over $16\sqrt{3}$ of containing P, and multiplying this by 4 gives us our expected value of

$$2 - \frac{8\pi}{9\sqrt{3}}$$

in this case. Finally, summing our expected values results in $11 - \frac{17\pi}{9\sqrt{3}}$, which extracts to $\boxed{040}$.